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DEPARTURES FROM MANY QUEUES IN SERIES

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Abstract

We consider a series of n single-server queues, each with unlimited waiting space and the first-in first-out service discipline. Initially, the system is empty; then k_n customers are placed in the first queue. The service times of all the customers at all the queues are i.i.d. with a general distribution. We show, under appropriate conditions, that the departure process from the n^{th} queue satisfies an invariance principle, converging as $n \to \infty$ after normalization to a reflected Brownian motion (RBM). As a consequence, the first k interdeparture times from queue n after the first departure are of order $\sqrt{n}$ as $n \to \infty$. Moreover, we establish a strong approximation showing that the maximum error in the RBM approximation for $k_n = n^a$ with $a > 0$ is of order $n^{(a-1/4)} \log n$. We then apply the subadditive ergodic theorem to the limiting RBM to show that the average of the first $\lfloor xn^a \rfloor$ interdeparture times from queue n after the first departure is of order $n^{(1-a)/2}$ for $0 \le a < 1$ and $x > 0$. Finally, we apply the subadditive ergodic theorem again to establish a hydrodynamic limit; i.e., we show that the average of the first $\lfloor xn \rfloor$ interdeparture time from queue n converges almost surely to a finite limit as $n \to \infty$ for each $x > 0$.

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Key words and phrases. tandem queues, queues in series, queueing networks, departure process, transient behavior, reflected Brownian motion, limit theorems, invariance principle, strong approximation, subadditive ergodic theorem, large deviations, interacting particle systems, hydrodynamic limit.

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Peter W. Glynn
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DEPARTMENT OF OPERATIONS RESEARCH
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1. Introduction and Summary

In this paper we consider a queueing model that could be used to represent the start-up behavior of a long production line or the transient flow of messages over a long path in a communication network. In particular, we consider a series of n single-server queues, each with unlimited waiting space and the first-in first-out service discipline. Initially, the system is empty; then k_n customers are placed in the first queue. The service times of all the customers at all the queues are i.i.d. with a general distribution having mean 1 and finite positive variance σ^2. Our object is to describe the departure process from the n^{th} queue as n gets large. (Equivalently, since customers are served in order of arrival, we can consider infinitely many queues in series with infinitely many customers in the first queue; we are still interested in the departure times of the first k_n customers from the n^{th} queue as $n \to \infty$.) We may have k_n constant, independent of n, or $k_n \to \infty$ as $n \to \infty$.

Let $D(k, n)$ be the departure time of customer k from queue n. Let $\Rightarrow$ denote convergence in distribution or weak convergence, as in Billingsley (1968). Since $D(1, n)$ is just the sum of n service times,

$$n^{-1/2}[D(1, n) - n] \Rightarrow N(0, \sigma^2) \text{ as } n \to \infty , \tag{1.1}$$

where $N(m, \sigma^2)$ denotes a normal random variable with mean m and variance σ^2. However, we are primarily interested in the interdeparture times between successive customers after the first from queue n. Let $\bar{D}(k, n)$ be the average of the first k interdeparture times from queue n after the first departure, i.e.,

$$\bar{D}(k, n) = [D(k + 1, n) - D(1, n)]/k . \tag{1.2}$$

Let $\lfloor x \rfloor$ denote the integer part of x. Here is what we regard as our main result.

Theorem 1.1. Suppose that the service times are i.i.d. with a general distribution having an exponential tail. Then, for $x > 0$ and $0 \le a \le 1$, there exists a proper random variable $L_a(x)$

such that

$$n^{-(1-a)/2}\bar{D}(\lfloor xn^a \rfloor, n) \Rightarrow L_a(x) \quad \text{as } n \to \infty . \tag{1.3}$$

Theorem 1.1 implies that the average of the first $\lfloor xn^a \rfloor$ interdeparture times from queue n after the first departure is asymptotically of order $n^{(1-a)/2}$ as n increases. A statement in terms of the departure times that can be shown to be equivalent to (1.3) is

$$n^{-(1+a)/2}[D(\lfloor xn^a \rfloor, n) - n] \Rightarrow L_a'(x) \text{ as } n \to \infty , \tag{1.4}$$

where $L_a'(x) = x L_a(x)$ for $0 < a \le 1$. (The relationship between $L_0'(x)$ and $L_0(x)$ is more complicated.)

Our primary focus is on the early departures from a large number of queues. For example, the customer index k_n associated with n queues might be k, $\sqrt{n}$ or n. However, there is a duality discussed in Section 2 that makes our results also applicable to a large number of departures from relatively few queues. In particular, under our i.i.d. assumption for the service times,

$$\{D(i, j) : 1 \le i \le k,\ 1 \le j \le n\} \stackrel{d}{=} \{D(j, i) : 1 \le j \le n,\ 1 \le i \le k\} \tag{1.5}$$

where $\stackrel{d}{=}$ denotes equality in distribution. Hence, associated with (1.3) and (1.4) are dual statements about the departure time of customer n from queue k_n. For example, as a consequence of (1.4), we obtain the following corollary.

Corollary. Under the conditions of Theorem 1.1, there exists a proper random variable $L_a^*(x)$ such that

$$n^{-(1+a)/2}(D(\lfloor xn^a \rfloor, n) - \lfloor xn^a \rfloor) \Rightarrow L_a^*(x) \text{ as } n \to \infty \tag{1.6}$$

for $x > 0$ and $a > 1$, where

$$L_a^*(x) = x^{-(1+a)/2} L'_{a^{-1}}(x^{-1/a}) , \tag{1.7}$$

so that

$$D(\lfloor xn^a \rfloor, n)/\lfloor xn^a \rfloor \Rightarrow 1 \quad \text{as } n \to \infty . \tag{1.8}$$

Three rather distinct cases are combined in Theorem 1.1: (i) $a = 0$, (ii) $0 < a < 1$ and (iii) $a = 1$. We treat the first case with $a = 0$ by establishing a functional central limit theorem (FCLT) for the departure process from queue n (Theorem 3.2); we treat the second case $0 < a < 1$ by establishing a strong approximation generalization of the FCLT and applying the subadditive ergodic theorem as on p. 277 of Liggett (1985) to the limiting RBM (Theorem 7.2); we treat the third case by directly applying the subadditive ergodic theorem (Theorem 6.3). In the first case with $a = 0$, $L_a(x)$ is a nondegenerate random variable, but in the other two cases $L_a(x)$ is deterministic. In the third case with $a = 1$, we establish almost sure convergence. Unfortunately, we do not yet know much about the limits in (1.3), (1.4) and (1.6). For $a < 1$, the limit $L_a(x)$ depends on the service-time distribution only through its first two moments. For $a = 1$, we conjecture that $L_a(x)$ depends on the service-time distribution beyond its first two moments.

This paper was largely motivated by Srinivasan (1989), who applied results of Rost (1981), Andjel (1982), Andjel and Kipnis (1984), Kipnis (1986), and Benassi and Fouque (1987) about interacting particle systems (in particular, the zero-range process and the asymmetric simple exclusion process) to describe the *hydrodynamic limit* for our model in the special case of *exponential service times* (still with mean 1). Roughly speaking, the hydrodynamic limit says that the average queue length among the first $[xt]$ queues at time t is asymptotically almost surely (a.s.) equal to $(2 - \sqrt{x})/\sqrt{x}$ as $t \to \infty$. Consequently, the average queue length among queues in the neighborhood of queue $[xt]$ is asymptotically a.s. $(1 - \sqrt{x})/\sqrt{x}$ as $t \to \infty$. (Note that the total number of customers in the first $\lfloor xt \rfloor$ queues is

$(2\sqrt{x} - x)t + o(t)$; then differentiate with respect to x. In the unsaturated case with external arrival process having rate $\lambda < 1$, asymptotically a.s. the first $(1 - \lambda)^2 t$ queues reach equilibrium at time t as $t \to \infty$, but the rest of the density profile remains the same.)

It is easy to apply Srinivasan's hydrodynamic limit in the saturated case (with i.i.d. exponential service times having mean 1) to deduce that the departure time of customer $\lfloor xn \rfloor$ from queue n is asymptotically a.s. $(1 + \sqrt{x})^2 n + o(n)$; see Section 6. Thus the departure times of customers 1 and n from queue n are a.s. $n + o(n)$ and $4n + o(n)$, respectively. To put this result in perspective, if customer n only had to wait at the first queue (as would be the case if all queues after the first had infinitely many servers), then the departure time for customer n from queue n would be a.s. $2n + o(n)$. Hence, the additional delay experienced by customer n in the last $n - 1$ queues is approximately equal to his delay in the first queue plus the sum of his service times.

Our case (iii) of Theorem 1.1 with $a = 1$ extends Srinivasan (1989) by establishing a hydrodynamic limit for *general service-time distributions*. As suggested by the discussion above, limits for the average queue length among the first $\lfloor xt \rfloor$ queues at time t as $t \to \infty$ are equivalent to limits for $n^{-1} D(\lfloor xn \rfloor, n)$ as $n \to \infty$, so case (iii) of Theorem 1.1 yields a hydrodynamic limit in the sense of Srinivasan (1989) for general service-time distributions. With regard to the interacting particle system literature, our result is interesting because the associated vector queue-length process depicting the number of customers at each queue (including the one in service, if any) is not Markov here. We treat this case by applying the subadditive ergodic theorem together with an upper bound based on a stochastic comparison involving associated random variables and the Cramer (1938)–Chernoff (1952) theorem about large deviations, e.g., see Vanderbei and Weiss (1988) or pp. 3.7 of Varadhan (1984). However, we have not yet identified the limit for general service-time distributions.

We also complement Srinivasan (1989) by describing in more detail what happens at the front of the ''wave'' of customers passing through the network. Of course, the first customer departs from queue n at time n with a deviation of order $\sqrt{n}$, as indicated in (1.1). The first case of Theorem 1.1 with $a = 0$ reveals that the first k interdeparture times from queue n after the first departure are each asymptotically of order $\sqrt{n}$ as $n \to \infty$. Consequently, by the time customer k has reached queue n for large n, customer k rarely has to wait. We treat this case by showing under appropriate conditions, that the departure process from the n^{th} queue obeys an invariance principle or functional central limit theorem (FCLT). The FCLT supports approximating the beginning of the departure process, after appropriate normalization, by an infinite-dimensional reflected Brownian motion (RBM) on the infinite-dimensional orthant $[0, \infty)^{\infty}$. This infinite-dimensional RBM is the natural extension of finite-dimensional RBMs considered by Harrison (1978), Harrison and Reiman (1981a,b), Reiman (1984) and Harrison and Williams (1987a,b).

The invariance principle implies that the approximation depends on the service-time distribution only through its mean and variance. Moreover, the mean and variance play a relatively trivial role. In particular, the mean service time only determines the deterministic rate customers flow through the queues; without loss of generality, we can let the mean service time be one, and we do. The service-time variance only appears (via its square root) as a constant multiplicative factor in front of the multivariate RBM associated with service-time variance 1. Hence, just as with the familiar one-dimensional Brownian motion (BM) approximation for partial sums of i.i.d. real-valued random variables, there is essentially only one fundamental limit process for this system for all service-time distributions. We call this limit process the *departure RBM*.

The model we consider has no external arrival process, but the same model can be interpreted as starting out empty with an external arrival process. Simply interpret the departure process from the first queue as the external arrival process. Of course, the assumption that the service times be all i.i.d. implies that the interarrival-time distribution must then be exactly the same as each service-time distribution. However, this is not required for the FCLT. The FCLT remains unchanged if the service-time distributions at an initial finite set of queues are different. (The stated results cover this generalization.)

To obtain further insight into the first appearance of congestion, we establish a strong approximation that shows that the error in the RBM approximation for the first $\lfloor xn^a \rfloor$ customers is $O(n^{(a-1/2)} \log n)$. We also show that the error in the RBM approximation is asymptotically negligible compared to the size of the first $\lfloor xn^a \rfloor$ components of the RBM. To obtain this result we also need to describe how the components of the departure RBM grow. Let $\hat{D}_k(1)$ denote the k^{th} coordinate of the departure RBM at time 1. We apply the subadditive ergodic theorem again to show that

$$k^{-1/2} \hat{D}_k(1) \Rightarrow L_1(1) \text{ as } k \to \infty \tag{1.9}$$

for $L_1(1)$ in (1.3); see Theorem 7.1. We thus obtain Theorem 1.1 for $0 < a < 1$; see Theorem 7.2.

The rest of this paper is organized as follows. In Section 2 we review a convenient representation for the departure process that facilitates its study. In particular, we exploit the fact that the departure time of customer k from queue n can be represented as the maximum partial sum of service times along nondecreasing paths of length $k + n - 1$ in a $k \times n$ lattice of service times. From this representation, the duality mentioned above is immediate.

In Section 3 we establish the FCLT and in Section 4 we establish the strong approximation needed for case (ii) of Theorem 1.1 with $0 < a < 1$. In Section 5 we establish stochastic order relations among the interdeparture times, which are of interest in their own right, but also help us describe the departure RBM and treat the case $k_n = O(n)$ in Section 6. In Section 6 we obtain our hydrodynamic limit, i.e., we treat the case $k_n = \lfloor xn \rfloor$. In Section 7 we establish (1.9) and the third case of Theorem 1.1. Finally, in Section 8 we make some concluding remarks.

We end this introduction by mentioning some additional references that provide background or treat somewhat related problems: Chapter 6 of Disney and Kiessler (1987), Kelly (1982, 1984), Suresh and Whitt (1990) and Vere-Jones (1968).

2. The Basic Recursion for the Departure Epochs

Let $V(k, n)$ be the service time and $D(k, n)$ the departure time for customer k at queue n. Our starting point is a basic recursion for the departure times,

$$D(k, n) = \max\{D(k-1), n), D(k, n-1)\} + V(k, n) \qquad (2.1)$$

for $k \geq 1$ and $n \geq 1$, with $D(k, 0) = 0$ for all k and $D(0, n) = 0$ for all n, which can be taken as the definition. (At this point, we do not assume that the service times are i.i.d.)

We can easily express $D(k, n)$ more directly in terms of the service times. To do so, let $\Pi(k, n)$ be the set of all "nondecreasing continuous paths" of length $k + n - 1$ from (1,1) to (k, n) in the set of ordered pairs $\mathcal{P} \equiv \{(i, j) : 1 \leq i \leq k,\ 1 \leq j \leq n\}$; i.e., $\pi \in \Pi(k, n)$ if π is a subset of $\mathcal{P}$ of cardinality $k + n - 1$ containing (1,1) and either $(i + 1, j)$ or $(i, j + 1)$, but not both, whenever it contains (i, j). Since successive ordered pairs in any such path π increase in the first component exactly $k - 1$ times, there are $\binom{k+n-2}{k-1}$ paths in $\Pi(k, n)$.

From (2.1), we easily establish the following by induction.

Proposition 2.1. For all $k \geq 1$ and $n \geq 1$,

$$D(k, n) = \max_{1 \le l \le n} \{D(k-1, l) + \sum_{j=l}^{n} V(k, j)\} \tag{2.2}$$

$$= \max_{1 \le l \le k} \{D(l, n-1) + \sum_{i=l}^{k} V(i, n)\} \tag{2.3}$$

$$= \max_{\pi \in \Pi(k, n)} \{ \sum_{(i, j) \in \pi} V(i, j)\} . \tag{2.4}$$

Evidently Proposition 2.1 is quite well known; e.g., formulas (2.1) and (2.3) appear as (1), (2) and (16) of Tembe and Wolff (1974). A variant of (2.4) for queues without extra waiting space appears in Muth (1979). As Muth observes, (2.4) implies that the departure times $D(k, n)$ are unchanged if we reverse the order of the queues and the order of the service times at each queue. Let superscripts index different models.

Corollary 1. If $V^2(i, j) = V^1(k-i, n-j)$ for $1 \le i \le k$, $1 \le j \le n$, then $D^2(k, n) = D^1(k, n)$.

Formula (2.4) also implies a certain duality, i.e., symmetry in k and n. Let $\stackrel{d}{=}$ denote equality in distribution.

Corollary 2. If $\{V^1(i, j) : 1 \le i \le k,\ 1 \le j \le n\} \stackrel{d}{=} \{V^2(j, i) : 1 \le i \le k,\ 1 \le j \le n\}$, then

$$\{D^1(i, j) : 1 \le i \le k,\ 1 \le j \le n\} \stackrel{d}{=} \{D^2(j, i) : 1 \le i \le k,\ 1 \le j \le n\} .$$

As an immediate consequence of Corollary 2, we obtain the following result in the i.i.d. setting which is of primary interest to us.

Corollary 3. If $V(i, j)$, $1 \le i \le k$, $1 \le j \le n$, are i.i.d., then (1.5) holds.

Corollaries 2 and 3 can be used to obtain limit theorems as $k \to \infty$ for fixed n from the limit theorems we establish as $n \to \infty$ for fixed k. Corollaries 2 and 3 also allow us to relate the interdeparture times of primary interest to us to associated sojourn times. The k^{th} *interdeparture time* from queue n is

$$\Delta(k, n) = D(k+1, n) - D(k, n) \tag{2.5}$$

with $D(0, n) = 0$, for $k \geq 0$ and $n \geq 1$. The *sojourn time* of customer k at queue n is

$$S(k, n) = D(k, n) - D(k, n-1) . \tag{2.6}$$

Corollary 4. Under the assumption of Corollary 3,

$$\{\Delta(i, j) : 0 \leq i \leq k-1,\ 1 \leq j \leq n\} \stackrel{d}{=} S(j, i) : 1 \leq j \leq n,\ 1 \leq i \leq k\} \tag{2.7}$$

Remarks. (2.1) Corollaries 3 and 4 immediately provide an analog of Theorem 1.1 for the average sojourn time of the first n customers from the first $\lfloor xn^a \rfloor$ queues after the first queue.

(2.2) The function mapping $\{V(i, j) : 1 \leq i \leq k,\ 1 \leq j \leq n\}$ into $\{D(i, j) : 1 \leq i \leq k,$ $1 \leq j \leq n\}$ is obviously nondecreasing and convex, so that stochastic order relations for service times carry over to departure times; see Stoyan (1983). The function is also Lipschitz, i.e., for each path π

$$\left| \sum_{(i,j) \in \pi} V^1(i, j) - \sum_{(i,j) \in \pi} V^2(i, j) \right| \leq \sum_{(i,j) \in \pi} \left| V^1(i, j) - V^2(i, j) \right|$$

and

$$\max_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \{ \left| D^1(i, j) - D^2(i, j) \right| \leq (k + n - 1) \max_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \{ |V^1(i, j) - V^2(i, j)| \}$$

so that there is model stability; see Whitt (1974).

3. The Functional Central Limit Theorem

We now apply (2.2) to show that $\{D(k, n)\}$ satisfies a FCLT as $n \to \infty$ when $\{V(k, n)\}$ does. (We do not assume that $\{V(k, n)\}$ is i.i.d. here.) For this purpose, let $D[0, \infty)$ be the space of right-continuous real-valued functions on the interval $[0, \infty)$ with limits from the left, endowed with the usual Skorohod (1956) J_1 topology; see Ethier and Kurtz (1986) or

Whitt (1980). Let $D[0, \infty)^{\infty}$ be the product space endowed with the product topology.

Let V_n and D_n be random elements of $D[0, \infty)^{\infty}$ defined as follows:

$$
\begin{aligned}
V_n &= (V_{1n}, V_{2n}, \ldots) \\
D_n &= (D_{1n}, D_{2n}, \ldots) \\
V_{kn}(t) &= n^{-\alpha}\left[\sum_{j=1}^{[nt]} V(k, j) - nt\right], \ t \geq 0 \ , \\
D_{kn}(t) &= n^{-\alpha}(D(k, [nt]) - nt) \ , \ t \geq 0 \ ,
\end{aligned}
\tag{3.1}
$$

for $\alpha > 0$.

Theorem 3.1. If $V_n \Rightarrow \hat{V}$ in $D[0, \infty)^{\infty}$ as $n \to \infty$ where $\hat{V}$ has continuous paths w.p.1, then $D_n \Rightarrow \hat{D}$ in $D[0, \infty)^{\infty}$ as $n \to \infty$, where $\hat{D} = f(\hat{V})$ with $f : D[0, \infty)^{\infty} \to D[0, \infty)^{\infty}$ defined by

$$f_1(x)(t) = x_1(t)$$

and

$$
\begin{aligned}
f_k(x)(t) &= \sup_{0 \leq s \leq t} \{f_{k-1}(s) + x_k(t) - x_k(s)\} \\
&= x_k(t) - \inf_{0 \leq s \leq t} \{x_k(s) - f_{k-1}(s)\}
\end{aligned}
\tag{3.2}
$$

for all $k \geq 2$ and $t \geq 0$.

Proof. First, from (2.2) and (3.1) it is immediate that $D_{1n} = V_{1n}$. Next,

$$
\begin{aligned}
D_{kn}(t) &= n^{-\alpha}(D(k, [nt] - nt) \\
&= n^{-\alpha}\left[\max_{1 \leq l \leq [nt]} \{D(k-1, l) + \sum_{j=l}^{[nt]} V(k, j)\} - nt\right] \\
&= n^{-\alpha}\left[\sup_{0 \leq s \leq t} \{D(k-1, [ns]) - ns + \sum_{j=[ns]}^{[nt]} V(k, j) - n(t-s)\}\right. \\
&= \sup_{0 \leq s \leq t} \{D_{k-1,n}(s) + V_{nk}(t) - V_{kn}(s) + n^{-\alpha} V(k, [ns])\} \ .
\end{aligned}
$$

However, since $V_{kn} \Rightarrow \hat{V}_k$ where $\hat{V}_k$ has continuous paths,

$$\sup_{0 \le s \le t} n^{-\alpha} V(k, [ns]) \Rightarrow 0 \quad \text{in} \quad D[0, \infty) ;$$

i.e., the maximum jump functional is continuous. Hence, by the convergence-together theorem (Theorem 4.1 of Billingsley) and induction, $(D_{1n}, \ldots, D_{kn})$ converges if $(f_1(V_n), \ldots, f_k(V_n))$ converges. However, it is easy to see (e.g., by Section 6 of Whitt (1980) and induction) that $(f_1, \ldots, f_k) : D[0, \infty)^\infty \to D[0, \infty)^k$ is continuous for each k. Since we are using the product topology, that implies that f itself is continuous. Hence, the desired convergence holds by the continuous mapping theorem (Theorem 5.1 of Billingsley). ■.

Remark (3.1) By the duality in Corollaries 2-4, Theorem 3.1 can also be regarded as a direct consequence of previous heavy-traffic limit theorems for the sojourn times of the first $\lfloor nt \rfloor$ customers at the first k queues; see Iglehart and Whitt (1970), Harrison (1973), and Reiman (1984). For the sojourn times, the case we consider corresponds to having the traffic intensity at queue i be $\rho_i = 1$ for all i. As in previous heavy-traffic limit theorems, we could let the service-time distributions change in the limit. ■

We can obtain a representation for the limit process $\hat{D}$ in Theorem 3.1 paralleling the representation of $D(k, n)$ as the maximal partial sum of the service times over all paths in $\Pi(k, n)$ in (2.4). For this purpose let $T_k(t)$ be the set of nondecreasing $(k+1)$-tuples $(t_0, t_1, \ldots, t_k)$ with $t_0 = 0$ and $t_k = t$. The following is deduced from (3.2) by induction on k.

Corollary. The limit process $\hat{D} \equiv \{\hat{D}_k : k \ge 1\} \equiv f(\hat{V}) \equiv \{f_k(\hat{V}_1, \ldots, \hat{V}_k) : k \ge 1\}$ can be represented as

$$\hat{D}_k(t) = \sup\{ \sum_{i=1}^{k} [\hat{V}_i(t_i) - \hat{V}_i(t_{i-1})] : (t_0, t_1, \ldots, t_k) \in T_k(t) \} \tag{3.3}$$

for all $k \ge 1$.

The standard case has normalization exponent $\alpha = 1/2$ in (3.1) and service-time limit process $\hat{V}$ being Brownian motion (BM), i.e., a vector of independent one-dimensional BMs. The resulting limit process $\hat{D}$ for the departure process is then an infinite-dimensional reflected Brownian motion (RBM) on the infinite-dimensional orthant. Such infinite-dimensional RBMs can be constructed by extending corresponding k-dimensional RBMs on the k-dimensional orthant; see p. 83 of Neveu (1965). The k-dimensional RBMs in turn coincide with those considered by Harrison (1978), Harrison and Reiman (1981a,b), Reiman (1984) and Harrison and Williams (1987a,b).

Let $\hat{B} = (\hat{B}_1, \hat{B}_2, \ldots)$ be a standard BM on $D[0, \infty)^\infty$, by which we mean a vector of independent standard (drift 0, diffusion coefficient 1) BMs. To obtain the standard limiting case, we assume that the service times are i.i.d. However, in order to cover the case of a general external arrival process, we exclude finitely many queues in the condition.

Theorem 3.2. If there exists a finite m such that $\{V(k, n) : k \geq 1, n \geq m\}$ is i.i.d. with $E\,V(1, m) = 1$ and $\mathrm{Var}\,V(1, m) = \sigma^2 < \infty$, then the condition of Theorem 3.1 holds with $\hat{V} = \sigma \hat{B}$ where $\hat{B}$ is a standard BM. Then $\hat{D} = \sigma f(\hat{B})$ for f in (3.2). The associated interdeparture-time limit process $\hat{\Delta}$, defined by $\hat{\Delta}_k = \hat{D}_{k+1} - \hat{D}_k$, $k \geq 1$, and $\hat{\Delta}_0 = D_1$, can be represented as

$$\hat{\Delta}_0 = \sigma \hat{B}_1 \ , \ \hat{Y}_k = \sigma \hat{B}_{k+1} - \sum_{i=0}^{k-1} \hat{\Delta}_i$$

$$\hat{\Delta}_k(t) = \hat{Y}_k(t) - \inf_{0 \leq s \leq t} \hat{Y}_k(s) \equiv \hat{Y}_k(t) + \hat{I}_k(t) \ , \ k \geq 1 \ . \tag{3.4}$$

Then $[(\hat{\Delta}_1, \ldots, \hat{\Delta}_k), (\hat{I}_1, \ldots, \hat{I}_k)]$ are the unique pair of k-dimensional processes so that $\hat{\Delta}_i(t) = \hat{Y}_i(t) + \hat{I}_i(t)$, $\hat{\Delta}_i(t) \geq 0$, $\hat{I}_i(t)$ is nondecreasing with $\hat{I}_i(0) = 0$ and

$$\int_0^t 1_{\{\hat{\Delta}_i(s) \neq 0\}} d\hat{I}_i(s) = 0$$

for $1 \leq i \leq k$ and $t \geq 0$. Moreover, for each k, $(\hat{\Delta}_1, \ldots, \hat{\Delta}_k)$ is a k-dimensional RBM as in

Harrison and Reiman (1981a,b) generated by a zero-drift BM with covariance matrix Σ having elements $\Sigma_{ii} = 2\sigma^2$, $1 \le i \le k$, $\Sigma_{i,i+1} = \Sigma_{i+1,i} = -\sigma^2$, $1 \le i \le k-1$, and $\Sigma_{ij} = 0$ otherw e, and reflection matrix $R = I - Q$, where $Q_{i,i+1} = 1$ for $1 \le i \le k-1$ and $Q_{ij} = 0$ otherwise.

Proof. By Theorems 3.2, 4.1 and 16.1 of Billingsley (1968), $V_n \Rightarrow \sigma\hat{B}$. By induction, $f(\sigma x) = \sigma f(x)$ for f in (3.2). Hence, $\hat{D} = f(\sigma\hat{B}) = \sigma f(\hat{B})$. The representatio.. (3.4) is an easy consequence of (3.2). The characterization of the pair $[(\hat{\Delta}_1, \ldots, \hat{\Delta}_k), (\hat{I}_1, \ldots, \hat{I}_k)]$ follows from repeated application of the one-dimensional characterization of the reflection map on p. 19 of Harrison (1985) (sometimes called Skorohod's lemma (1961)), and induction. The characterization of $(\hat{\Delta}_1, \ldots, \hat{\Delta}_k)$ as an RBM follows by the arguments of Harrison (1978) and Harrison and Reiman (1981a,b) or directly from those papers, after exploiting the duality in Corollaries 2-4 of Proposition 2.1. The RBM structure is easy to see in this case of an acyclic network by writing (3.4) in differential form. Then

$$\begin{aligned} d\hat{\Delta}_0 &= d\hat{B}_1 \\ d\hat{\Delta}_k &= d\hat{B}_{k+1} - \sum_{i=0}^{k-1} d\hat{\Delta}_i + d\hat{I}_k \, . \end{aligned} \tag{3.5}$$

By induction, (3.5) can be rewritten as

$$\begin{aligned} d\hat{\Delta}_0 &= d\hat{B}_1 \\ d\hat{\Delta}_1 &= d\hat{B}_2 - d\hat{B}_1 + d\hat{I}_1 \\ d\hat{\Delta}_k &= d\hat{B}_{k+1} - d\hat{B}_k - d\hat{I}_{k-1} + d\hat{I}_k \, , \; k \ge 2 \, . \end{aligned} \tag{3.6}$$

This is the differential form for the RBM plus Δ_0; i.e., from (3.6) we obtain $Z = X + YR$ as in Harrison and Reiman (1981a,b), where $Z = (\hat{\Delta}_1, \ldots, \Delta_k)$, X is the BM with components $X_i = \hat{B}_{i+1} - \hat{B}_i$ and $Y = (\hat{I}_1, \ldots, \hat{I}_k)$. ■

Remarks (3.2) Additional characterizations of the departure RBM such as the generator and a generalized Itô's formula follow from Harrison and Reiman (1981a,b). Since the BMs $\hat{B}_i$ in the construction have zero drift, the departure RBM does not have a proper stationary distribution.

(3.3) We do not know much about the joint distribution of $(\hat{\Delta}_1(1), \ldots, \hat{\Delta}_k(1))$. However, since $\hat{\Delta}_1 = \sigma\hat{B}_2 - \sigma\hat{B}_1$, $\hat{\Delta}_1 \stackrel{d}{=} \sqrt{2}\,\sigma\,|B_1|$. Hence, $\hat{\Delta}_1(1)$ has a positive normal distribution with $E[\hat{\Delta}_1(1)] = 2\sigma/\sqrt{\pi}$ and $E[\hat{\Delta}_1(1)^2] = 2\sigma^2$. In Section 5 we show that $\hat{\Delta}_k(t)$ is stochastically decreasing in k and stochastically increasing in t.

4. The Strong Approximation

Under the assumptions of Theorem 3.2, we know that the interdeparture times of the k^{th} customer from the n^{th} queue are asymptotically of order $\sqrt{n}$ as $n \to \infty$ for any k. We now want to say what happens if the customer index increases with n. For this purpose, we establish a strong approximation result, drawing on Komlós, Major and Tusnády (1975, 1976); see p. 107 of Csörgő and Révész (1981). We show that the error in the diffusion approximation is $O(n^{(a - \frac{1}{2})}\log n)$ when the largest customer index k is n^a. We state the result below in an equivalent unnormalized form; to obtain the stated bound, divide through by $\sqrt{n}$.

Theorem 4.1. If, in addition to the assumption of Theorem 3.2, all service times are independent and there exist positive constants K and λ such that $P(V(k, j) > x) \leq Ke^{-\lambda x}$ for all k, j and x, then there exists a probability space supporting the departure times $D(k, j)$ and the limit process $\hat{D} = \sigma f(\hat{B})$ such that, for any $a > 0$,

$$\max_{\substack{1 \leq k \leq \lfloor n^a \rfloor \\ 1 \leq j \leq n}} \{|D(k, j) - j - \sqrt{n}\,\hat{D}_k(j/n)|\} = O(n^a \log n) \text{ a.s.}$$

Remarks. (4.1) Theorem 4.1 establishes part of the second case of Theorem 1.1 with $0 < a < 1$. To determine the order of magnitude of $D(k_n, n)$ for $k_n = \lfloor xn^a \rfloor$ for $0 < a < 1$, we have thus reduced the problem to determining how $\hat{D}_k(1)$ behaves as $k \to \infty$, which we discuss in Section 7.

(4.2) In Theorem 4.1 we focus on the departure times, but a corresponding result holds for the interdeparture times $\Delta(k, n)$ in (2.5) by applying the triangle inequality. In particular, as an immediate consequence of Theorem 4.1,

$$\max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le j \le n}} \{ | \Delta(k, j) - \sqrt{n} [\hat{D}_{k+1}(j/n) - \hat{D}_k(j/n)] | \} = O(n^a \log n) \text{ a.s.} \qquad (4.1)$$

Theorem 4.1 is proved by combining Lemmas 4.4 and 4.5 below. Lemmas 4.1–4.3 below are used to prove Lemma 4.4.

Lemma 4.1. If $\{U_k : k \ge 1\}$ is a sequence of independent random variables and there exist positive constants K and λ such that $P(U_k > x) \le Ke^{-\lambda x}$ for all $x > 0$, then for any $a > 0$

$$\max_{1 \le k \le \lfloor n^a \rfloor} \{U_k\} = O(\log n) \quad \text{a.s.}$$

Proof. For any x_n,

$$P\left\{ \max_{1 \le k \le \lfloor n^a \rfloor} \{U_k\} > x_n \right\} \le 1 - (1 - Ke^{-\lambda x_n})^{n^a} .$$

Hence, for $x_n = (a + 2)\log n/\lambda$,

$$\begin{aligned} P(A_n) \equiv P(\max_{1 \le k \le \lfloor n^a \rfloor} \{U_k\} > \frac{a+2}{\lambda} \log n) &\le 1 - (1 - Kn^{-(a+2)})^{n^a} \\ &\le 1 - \exp(\log[(1 - Kn^{-(a+2)})^{n^a}]) \\ &\le 1 - \exp(n^a \log[1 - Kn^{-(a+2)}]) \\ &\le - n^a \log(1 - Kn^{-(a+2)}) \\ &\le 2Kn^{-2} \quad \text{for } n \text{ sufficiently large} \end{aligned}$$

using $e^{-x} \ge 1 - x$ in the second to last step and $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$ for $0 < x < 1$ in the last step. Since $\sum_{n=1}^{\infty} P(A_n) < \infty$, $P(A_n$ infinitely often$) = 0$ by the

Borel-Cantelli lemma. Hence, there are positive random variables X_1 and X_2 such that

$$\max_{1 \le k \le \lfloor n^a \rfloor} \{U_k\} \le X_1 + X_2 \log n \quad \text{for all } n \ge 1 \quad \text{a.s.} \qquad \blacksquare$$

We now extend a strong approximation result of Komlós, Major and Tusnády (1975, 1976), p. 107 of Csörgő and Révész (1981).

Lemma 4.2. Under the assumptions of Theorem 4.1, there is a probability space supporting independent standard BMs $\hat{B}_k$ and the service times so that

$$\max_{\substack{1 \le k < \lfloor n^a \rfloor \\ 1 \le l \le n}} \left\{ \left| \sum_{j=1}^{l} V(k, j) - l - \sigma \hat{B}_k(l) \right| \right\} = O(\log n) \quad \text{a.s.}$$

Proof. The service times of all customers at all queues after the first m are i.i.d., but we do not have identical distributions at earlier queues. However, by Lemma 4.1 and the assumption of Theorem 4.1, without loss of generality it suffices to assume that all the service times are i.i.d. To support half this claim, note that $\sigma \hat{B}_k(l)$ is normally distributed with mean 0 and variance $l\sigma^2$, so that these variables satisfy the same tail condition imposed on the service times for $1 \le l \le m$ (possibly with different constants K and λ). Hence, it suffices to assume that $\{V(k, j)\}$ is i.i.d., with the distribution of $V(1, m)$, and we do. By Komlós, Major and Tusnády (1975, 1976), for each k there is a probability space containing a BM $\hat{B}_k$ such that

$$P\{ \max_{1 \le l \le n} | \sum_{j=1}^{l} V(k, j) - l - \sigma \hat{B}_k(l) | > C \log n + x \} < K e^{-\lambda x}$$

for positive constants C, K and λ depending on the distribution of $V(1, m)$. Hence, using a product space, we can achieve

$$P(A_n) \equiv P\left\{ \max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le l \le n}} \left| \sum_{j=1}^{l} V(k, j) - l - \sigma \hat{B}_k(l) \right| > C \log n + x_n \right\}$$

$$\le 1 - (1 - Ke^{-\lambda x_n})^{\lfloor n^a \rfloor} .$$

As in Lemma 4.1, choose $x_n = (a + 2)\log n/\lambda$ to obtain $P(A_n) \le 2Kn^{-2}$ for n sufficiently large. By Borel-Cantelli, $P(A_n$ infinitely often$) = 0$. Hence, there exist random variables X_1 and X_2 such that

$$\max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le l \le n}} \left\{ \left| \sum_{j=1}^{l} V(k, j) - l - \sigma \hat{B}_k(l) \right| \right\} \le X_1 + X_2 \log n \quad \text{a.s.} \quad \blacksquare$$

For the next lemma, we specify some quantities associated with a real-valued function defined on the positive integers, say y. Let

$$y^{\uparrow}(n) = \max_{1 \le k \le n} y(k) \quad \text{and} \quad \|y\|_n = |y|^{\uparrow}(n) \ , \ n \ge 1 \ . \tag{4.2}$$

The following elementary lemma can be viewed as a special case of Theorem 6.1 of Whitt (1980).

Lemma 4.3. For all $n \ge 1$, $\|y_1^{\uparrow} - y_2^{\uparrow}\|_n \le \|y_1 - y_2\|_n$.

Let $D^*(k, n)$ be the following function of the limiting BM $\hat{B}$,

$$\begin{aligned} D^*(1, n) &= \sigma \hat{B}_1(n) \\ D^*(k, n) &= \sigma \hat{B}_k(n) - \min_{1 \le j \le n} \{\sigma \hat{B}_k(j) - D^*(k-1, j)\} \\ &= \max_{1 \le j \le n} \{D^*(k-1, j) + \sigma \hat{B}_k(n) - \sigma \hat{B}_k(j)\} \end{aligned} \tag{4.3}$$

for $n \ge 1$ and $k \ge 2$. Let e denote the identity function, i.e., $e(t) = t, t \ge 0$.

Lemma 4.4. Under the assumptions of Theorem 4.1, for any $a > 0$ there exists a probability space supporting the departure times $D(k, j)$ and the process D^* in (4.3) such that

$$\max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le j \le n}} \{|D(k, j) - j - D^*(k, j)|\} = O(n^a \log n) \text{ a.s.}$$

Proof. Note that (4.3) is not quite the same function of $\sigma \hat{B}_k(n)$ as $D(k, n)$ is of $\sum_{j=1}^{n} [V(k, j) - j]$ in (2.2). The exactly corresponding function is

$$D'(1, n) = \sigma \hat{B}_1(n)$$
$$D'(k, n) = \max_{1 \le j \le n} \{D(k-1, j) + \sigma \hat{B}_k(n) - \sigma \hat{B}_k(j-1)\} . \qquad (4.4)$$

However, by (2.2), (4.1), (4.2), (4.4) and Lemmas 4.1 and 4.3, there are random variables X_1 and X_2 such that

$$\begin{aligned} \|D'(k,\cdot) - D^*(k,\cdot)\|_n &\le \|D'(k-1,\cdot) - D^*(k-1,\cdot)\|_n \\ &\quad + \max_{1 \le j \le n} \{|\sigma \hat{B}_k(j) - \sigma \hat{B}_k(j-1)|\} \\ &\le \|D'(k-1,\cdot) - D^*(k-1,\cdot)\|_n + X_1 + X_2 \log n \qquad (4.5) \\ &\le k(X_1 + X_2 \log n) \quad \text{for } k \le \lfloor n^a \rfloor , \end{aligned}$$

where we have used the fact that $\sigma \hat{B}_k(j) - \sigma \hat{B}_k(j-1)$ are i.i.d. normal random variables in the second to last step and induction in the last step. In particular, by Lemma 4.1,

$$\max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le j \le n}} \{|\sigma \hat{B}_k(j) - \sigma \hat{B}_k(j-1)|\} = O(\log n) \text{ a.s.}$$

Hence,

$$\max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le j \le n}} \{|D'(k, j) - D^*(k, j)|\} \le n^a (X_1 + X_2 \log n)$$

and it suffices to do the proof with D' in (4.4) instead of D^* in (4.3). By an argument just like (4.5), using Lemma 4.2 now, there exist a probability space supporting the processes D and D' and finite random variables X_1 and X_2 such that

$\|D(1,\cdot) - e(\cdot) - D'(1,\cdot)\|_n \le X_1 + X_2 \log n$ a.s.

and

$\|D(k,\cdot) - e(\cdot) - D'(k,\cdot)\|_n \le X_1 + X_2 \log n + \|D(k-1,\cdot) - e(\cdot) - D'(k-1,\cdot)\|_n,$ a.s.

for $1 \le k \le \lfloor n^a \rfloor$. Hence,

$$\|D(k,\cdot) - e(\cdot) - D'(k,\cdot)\|_n \le \lfloor n^a \rfloor (X_1 + X_2 \log n) = O(n^a \log n) \quad \text{a.s.}$$

for all $k \le \lfloor n^a \rfloor$. ∎

To prove our next lemma we want a continuous analog of (4.2). For a real-valued function of a real-variable, say y, let

$$y^{\uparrow}(t) = \sup_{0 \le s \le t} y(s) \quad \text{and} \quad \|y\|_t = |y|^{\uparrow}(t),\ t \ge 0\ . \tag{4.6}$$

Paralleling Lemma 4.3,

$$\|y_1^{\uparrow} - y_2^{\uparrow}\|_t \le \|y_1 - y_2\|_t\ . \tag{4.7}$$

Lemma 4.5. For any $a > 0$

$$\max_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 1 \le j \le n}} \{|D^*(k,j) - \sqrt{n}\hat{D}_k(j/n)|\} = O(n^a \log n) \quad \text{a.s.}$$

Proof. Note that

$$\{\sqrt{n}\hat{D}(t/n) : t \ge 0\} \stackrel{d}{=} \{\hat{D}(t) : t \ge 0\}$$

and

$$\{D^*(k,[t]) : k \ge 1,\ t \ge 0\} \stackrel{d}{=} \{\hat{D}_k([t]) : k \ge 1,\ t \ge 0\}\ .$$

Hence, what we want to show is

$$\sup_{\substack{1 \le k \le \lfloor n^a \rfloor \\ 0 \le t \le n}} \{|\hat{D}_k(t) - \hat{D}_k([t])|\} = O(n^a \log n) \quad \text{a.s.} \tag{4.8}$$

By (4.7),

$$\|\hat{D}_k(\cdot) - \hat{D}_k([\cdot])\|_n \le \|\hat{D}_{k-1}(\cdot) - \hat{D}_{k-1}([\cdot])\|_n + \|\sigma\hat{B}_k(\cdot) - \sigma\hat{B}_k([\cdot])\|_n \ . \qquad (4.9)$$

However,

$$\max_{1 \le k \le \lfloor n^* \rfloor} \|\sigma\hat{B}_k(\cdot) - \sigma\hat{B}_k([\cdot])\|_n \le \sigma \max_{\substack{1 \le k \le \lfloor n^* \rfloor \\ 1 \le j \le n}} \{ \sup_{j < s < j+1} \{ |\hat{B}_k(s) - \hat{B}_k(j)| \} \} \qquad (4.10)$$

$$\le X_1 + X_2 \log n \quad \text{a.s.}$$

for finite random variables X_1 and X_2, by Lemma 4.1. Combining (4.9) and (4.10) gives (4.8). ■

5. Stochastic Order for the Interdeparture Times

In this section we establish stochastic comparisons for the interdeparture times $\Delta(k, n)$ in (2.5). We say that a random element X_1 is stochastically less than or equal to another random element X_2, and write $X_1 \le_{st} X_2$, if $Eh(X_1) \le Eh(X_2)$ for all nondecreasing bounded measurable real-valued functions h; see Kamae, Krengel and O'Brien (1977). We are interested in the case X_i is an array of real-valued random variables. As before, let $\overset{d}{=}$ denote equality in distribution.

Theorem 5.1. Suppose that the service times $V(k, n)$ are all mutually independent.

(a) If $V(k, n) \overset{d}{=} V(1, n)$ for all $k \ge 1$ and $n \ge 1$, then

$$\{\Delta(k+1, n) : k \ge 1, n \ge 1\} \le_{st} \{\Delta(k, n) : k \ge 1, n \ge 1\} ,$$

so that

$$V(1, n) \le_{st} \Delta(k+1, n) \le_{st} \Delta(k, n) \quad \text{for } k \ge 1 \text{ and } n \ge 1 \ .$$

(b) If $V(k, n) \overset{d}{=} V(k, 1)$ for all $k \ge 1$ and $n \ge 1$, then

$$\{\Delta(k, n) : k \ge 1, n \ge 1\} \le_{st} \{\Delta(k, n+1) : k \ge 1, n \ge 1\} ,$$

so that

$$V(k+1,1) \leq_{st} \Delta(k,n) \leq_{st} \Delta(k,n+1) \text{ for } k \geq 1 \text{ and } n \geq 1 .$$

Proof. We do only part (a) because the proof of (b) is similar. We construct a process $\{\tilde{\Delta}(k+1,n) : k \geq 1, n \geq 1\}$ with the same finite-dimensional distributions as $\{\Delta(k+1,n) : k \geq 1, n \geq 1\}$ such that

$$\tilde{\Delta}(k+1,n) \leq \Delta(k,n) \quad \text{a.s. for all } k \geq 1 \text{ and } n \geq 1 . \tag{5.1}$$

For this purpose, we use service times $\tilde{V}(k,n)$ defined by $\tilde{V}(k+1,n) = V(k,n)$ for all k and n. By our assumptions, $\{\tilde{V}(k,n) : k \geq 1, n \geq 1\}$ is distributed the same as $\{V(k,n) : k \geq 1, n \geq 1\}$. We define $\tilde{\Delta}(k,n)$ and $\tilde{D}(k,n)$ just like $\Delta(k,n)$ and $D(k,n)$ but using the service times $\tilde{V}(k,n)$ instead of $V(k,n)$.

By (2.1),

$$\begin{aligned} \Delta(k,n) &= \max\{D(k,n), D(k+1,n-1)\} + V(k+1,n) - D(k,n) \\ &= [D(k+1,n-1) - D(k,n)]^+ + V(k+1,n) . \end{aligned} \tag{5.2}$$

Hence,

$$\tilde{\Delta}(k+1,n) = [\tilde{D}(k+2,n-1) - \tilde{D}(k+1,n)]^+ + V(k,n) . \tag{5.3}$$

From (5.2) and (5.3), we see that (5.1) holds if

$$\tilde{D}(k+2,n-1) - \tilde{D}(k+1,n) \leq D(k+1,n-1) - D(k,n) \tag{5.4}$$

for all $k \geq 1$ and $n \geq 1$. We establish (5.4) by induction on the sum of the indices ($m = k + n$ in $D(k,n)$). Note that

$$\begin{aligned}
\tilde{D}(k+2, n-1) - \tilde{D}(k+1, n) = \\
&\max\{\tilde{D}(k+1, n-1), \tilde{D}(k+2, n-2)\} + \tilde{V}(k+2, n-1) \\
&- \max\{\tilde{D}(k, n), \tilde{D}(k+1, n-1)\} + \tilde{V}(k+1, n) \\
= &[\tilde{D}(k+2, n-2) - \tilde{D}(k+1, n-1)]^+ + \tilde{V}(k+2, n-1) \\
&- [\tilde{D}(k, n) - \tilde{D}(k+1, n-1)]^+ + \tilde{V}(k+1, n) \\
= &[\tilde{D}(k+2, n-2) - \tilde{D}(k+1, n-1)]^+ + V(k+1, n-1) \\
&+ [\tilde{D}(k+1, n-1) - \tilde{D}(k, n)]^- - V(k, n) \\
\le &[D(k+1, n-2) - D(k, n-1)]^+ + V(k+1, n-1) \\
&+ [D(k, n-1) - D(k-1, n)]^- - V(k, n) \equiv Z
\end{aligned}$$

by the induction hypothesis, where

$$\begin{aligned}
Z &= \max\{D(k+1, n-2), D(k, n-1)\} - D(k, n-1) + V(k+1, n-1) \\
&\quad - \max\{D(k-1, n), D(k, n-1)\} + D(k, n-1) - V(k, n) \\
&= D(k+1, n-1) - D(k, n) \ .
\end{aligned}$$

To start the induction, note that, for $n = 1$ and any k,

$$\begin{aligned}
\tilde{D}(k+2, n-1) - \tilde{D}(k+1, n) &= -\tilde{D}(k+1, 1) \\
= -(\tilde{V}(1, 1) + \cdots + \tilde{V}(k+1, 1)) &\stackrel{d}{=} -\tilde{V}(1, 1) - [V(1, 1) + \cdots + V(k, 1)] \\
&\le -D(k, 1) = D(k+1, 0) - D(k, 1) \ .
\end{aligned}$$

Hence, (5.4) is established and the proof is complete. ∎

Corollary 1. Under the conditions of Theorem 5.1(a), for each $n \ge 1$ there exists a proper stochastic process $\{\tilde{\Delta}(k, n) : k \ge 1\}$ with $\tilde{\Delta}(k, n) \ge_{st} V(1, n)$ such that

$$\{\Delta(k+j, n) : k \ge 1\} \Rightarrow \{\tilde{\Delta}(k, n) : k \ge 1\} \text{ in } R^\infty \text{ as } j \to \infty \ .$$

We can apply Theorem 5.1 to obtain a stochastic comparison for the limit process in Theorem 3.1. We actually focus on the associated interdeparture-time limit process $\hat{\Delta}_k(t) = \hat{D}_{k+1}(t) - \hat{D}_k(t)$.

Corollary 2. Suppose that the service times $V(k, n)$ are all independent and the FCLT $V_n \Rightarrow \hat{V}$ holds as required for Theorem 3.1.

(a) If $V(k, n) \stackrel{d}{=} V(1, n)$ for all $k \geq 1$ and $n \geq 1$, then

$$\{\hat{\Delta}_{k+1}(t) : k \geq 1, t \geq 0\} \leq_{st} \{\hat{\Delta}_k(t) : k \geq 1, t \geq 0\}$$

for all $k \geq 1$.

(b) If $V(k, n) \stackrel{d}{=} V(k, 1)$ for all $k \geq 1$ and $n \geq 1$, then

$$\{\hat{\Delta}_k(t) : k \geq 1, t \geq 0\} \leq_{st} \{\hat{\Delta}_k(t+u) : k \geq 1, t \geq 0\}$$

for all $u > 0$.

Proof. Use the fact that stochastic order is preserved under weak convergence. ■

6. The Hydrodynamic Limit: The Case $\mathbf{k_n = O(n)}$

In this section we describe the behavior of $D(k_n, n)$ (and, equivalently, $D(n, k_n)$) when k_n is of order n. We first apply the hydrodynamic limit of Rost (1981) as discussed in Section 4.2 of Srinivasan (1989) to treat the special case of exponential service times.

Theorem 6.1. If all the service times are i.i.d. with an exponential distribution having mean 1, then

$$\lim_{n \to \infty} n^{-1} D(\lfloor xn \rfloor, n) = (1 + \sqrt{x})^2 \quad \text{a.s.} \qquad \text{for any } x > 0 .$$

Proof. By Section 4.2 of Srinivasan (1989), the average queue length among the first $\lfloor xt \rfloor$ queues at time t is asymptotically a.s. $(2 - \sqrt{x})/\sqrt{x}$ as $t \to \infty$. Hence, for $x > 1$, the average queue length among the first n and $\lfloor x^2 n \rfloor$ queues at time $\lfloor x^2 n \rfloor$ are asymptotically a.s. $(2 - x^{-1})/x^{-1} = 2x - 1$ and 1, respectively, as $n \to \infty$. Hence asymptotically a.s. there are $x^2 n + o(n)$ customers in queues 2 through $x^2 n$ and $(2x - 1)n + o(n)$ customers in queues 2 through n. Hence, asymptotically a.s. $(x^2 - 2x + 1)n + o(n)$ customers have departed from queue n, and the departure time for customer $(x^2 - 2x + 1)n$ from queue n is $x^2 n + o(n)$. Now do a change of variables, replacing $(x - 1)^2$ by x.

To treat $x < 1$, note that $n^{-1} D(\lfloor x^2 n \rfloor, n) \stackrel{d}{=} n^{-1} D(n, \lfloor x^2 n \rfloor)$. Let $n' = x^2 n$. Then $n^{-1} D(n, \lfloor x^2 n \rfloor) = (x^2/n) D(\lfloor n/x^2 \rfloor, n) + o(1)$. From the previous argument,

$$(x^2/n) D(\lfloor n/x^2 \rfloor, n) \to x^2 \left[\frac{1}{x} + 1 \right]^2 = (x+1)^2 \quad \text{a.s. as } n \to \infty .$$

For $x = 1$, consider the average queue lengths among the first n and $4n$ queues, and reason similarly. ■

We now establish the existence of a limit for a general service-time distribution having an exponential tail. First, recall that under the conditions of Theorem 4.1,

$$\max_{\substack{1 \le i \le \lfloor xn \rfloor \\ 1 \le j \le n}} \{V(i, j)\} = O(\log n) \quad \text{a.s.}$$

by Lemma 4.1, so that

$$D(\lfloor xn \rfloor, n) \le O(n \log n) \quad \text{a.s.}$$

However, we will show that $D(\lfloor xn \rfloor, n)$ is actually $O(n)$.

For this purpose, we exploit a stochastic comparison involving associated random variables; see p. 29 of Barlow and Proschan (1975). Recall that a family of random variables are *associated* if all pairs of nondecreasing bounded real-valued functions of the random variables have nonnegative correlation.

Lemma 6.1. If the service times $\{V(i, j) : 1 \le i \le k,\ 1 \le j \le n\}$ are independent or just associated, then the partial sums $\sum_{(i,j) \in \pi} V(i, j)$ for the $\binom{k+n-2}{k-1}$ paths π in $\Pi(k, n)$ are associated random variables.

Proof. The partial sums are all nondecreasing functions of the kn service times. ■

Theorem 6.2. If the service times $V(i, j)$ are all independent, then

$$D(k, n) \leq_{st} \max\{S_\pi : \pi \in \Pi(k, n)\}$$

where S_π, $\pi \in \Pi(k, n)$, are mutually independent with

$$S_\pi \overset{d}{=} \sum_{(i, j) \in \pi} V(i, j)$$

for each path π.

Proof. Apply Theorem 3.2, p. 33 of Barlow and Proschan (1975). ■

We now use this stochastic bound to develop a bound and heuristic estimate for $\lim_{n \to \infty} n^{-1} D(\lfloor xn, n \rfloor)$ for a general service-time distribution. We call this the *path-independence bound.* We also use this bound to show that the limit exists.

Theorem 6.3. If all the service times are i.i.d. with $EV(1, 1) = 1$ and there exist positive constants K and λ such that $P(V(1, 1) > x) \leq Ke^{-\lambda x}$ for all $x > 0$, then there exists a deterministic strictly increasing concave function $\gamma(x)$ with $\gamma(x) \geq 1$, $\gamma(x + y) - \gamma(x) \geq y$ and $\gamma(x) = x\gamma(x^{-1})$ for all $x, y > 0$ such that

$$\lim_{n \to \infty} n^{-1} D(\lfloor nx \rfloor, n) = \gamma(x) \quad \text{a.s.} \tag{6.1}$$

and

$$\lim_{n \to \infty} E|n^{-1} D(\lfloor nx \rfloor, n) - \gamma(x)| = 0 \tag{6.2}$$

for all $x > 0$. Moreover,

$$\gamma(x) \geq \begin{cases} 1 + xE(\max\{V(1, 2), V(2, 1)\}) \geq 1 + x\,, & 0 < x \leq 1 \\ x + E(\max\{V(1, 2), V(2, 1)\}) \geq 1 + x\,, & 1 \leq x\,, \end{cases} \tag{6.3}$$

and

$$\gamma(x) \leq (1 + a^*)(1 + x)\,, \tag{6.4}$$

where

$$a^* \equiv a^*(x) = \inf\{a > 0 : (1 + x)h(a) > (1 + x) \log (1 + x) - x \log x\} \qquad (6.5)$$

and

$$h(a) = \sup_{\theta} \{\theta a - \log E e^{\theta[V(1,m) - 1]}\} . \qquad (6.6)$$

Proof. We first establish the upper bound in (6.4). Using Stirling's formula, p. 52 of Feller (1968), we see that the number of paths in $\Pi(\lfloor xn \rfloor, n)$ is $\phi(x, n) = e^{n\psi(x) + o(n)}$, where

$$\psi(x) = (1 + x) \log (1 + x) - x \log x .$$

Let π_n be a path in $\Pi(\lfloor xn \rfloor, n)$ and let S_{π_n} be the partial sum of all service times on path π_n. By the Cramér (1938)–Chernoff (1952) theorem,

$$P(S_{\pi_n} > (1 + a)(1 + x)n) = e^{-(1 + x)nh(a) + o(n)}$$

where h is defined in (6.6). Using the path-independence bound established in Theorem 6.2, we have

$$P(D(\lfloor xn \rfloor, n) > (1 + a)(1 + x)n) \le 1 - (1 - e^{-n(1 + x)h(a) + o(n)})^{\phi(x, n)} . \qquad (6.7)$$

The critical case $a = a^*(x)$ given by (6.5). If $h(a) > h(a^*)$, then the probability in (7.7) converges to 0, whereas if $h(a) < h(a^*)$, then the probability converges to 1. In particular, for any $a > a^*$,

$$\sum_{n=1}^{\infty} P(D(\lfloor xn \rfloor, n) > (1 + a)(1 + x)n) < \infty ,$$

so that we can apply Borel-Cantelli to deduce that

$$\overline{\lim_{n \to \infty}}\, n^{-1} D(\lfloor xn \rfloor, n) \le (1 + a)(1 + x) \quad \text{a.s.}$$

Hence, we have the claimed upper bound in (6.4).

Now we apply the subadditive ergodic theorem on p. 277 of Liggett (1985) to establish the existence of the limit. We first consider the limit of $n^{-1}D(kn, ln)$ for k and l integer. We let $X_{0,0} = 0$,

$$-X_{0,n} = D(kn, ln) - V(1, 1)$$

and $-X_{m,n}$ be $-X_{0,n-m}$ applied to the shifted service times $V'(i, j) = V(i + km, j + lm)$; e.g., for $k = l = 1$, $X_{n-1,n} = 0$ and $-X_{n-2,n} = V(n, n) + \max\{V(n-1, n), V(n, n-1)\}$. With this definition, X is subadditive, i.e., $X_{0,n} \leq X_{0,m} + X_{m,n}$ for $0 \leq m \leq n$. Moreover, X satisfies the other conditions of the subadditive ergodic theorem; in particular, $\{X_{(n-1)k,nk} : n \geq 1\}$ is a stationary process for each k, $\{X_{m,m+k} : k \geq 0\} \stackrel{d}{=} \{X_{m+1,m+k+1} : k \geq 0\}$ for each m, and $E(X_{01}^{+}) < \infty$. Hence, $n^{-1}D(kn, ln)$ has a deterministic limit $\eta(k, l)$ in the sense of (6.1) and (6.2) for all integers k and l. When n is a multiple of l,

$$\frac{1}{n} D\left(\frac{kn}{l}, n\right) = \frac{1}{l}\, \frac{l}{n} D\left(k\frac{n}{l}, l\frac{n}{l}\right) \to \frac{1}{l}\eta(k, l) \text{ as } n \to \infty \quad \text{a.s.}$$

More generally,

$$D\left(\frac{k}{l}\, l\lfloor n/l\rfloor, l\lfloor n/l\rfloor\right) \leq D(\lfloor kn/l\rfloor, n) \leq D\left(\frac{k}{l}\, l(\lfloor n/l\rfloor + 1), l([n/l] + 1)\right),$$

where

$$n^{-1} D\left(\frac{k}{l}\, l\lfloor n/l\rfloor, l\lfloor n/l\rfloor\right) = \frac{\lfloor n/l\rfloor}{n}\, \frac{1}{\lfloor n/l\rfloor} D(k\lfloor n/l\rfloor), l\lfloor n/l\rfloor) \to \frac{1}{l}\eta(k, l)$$

and

$$n^{-1} D\left[\frac{k}{l}\, l(\lfloor n/l \rfloor + 1),\, l(\lfloor n/l \rfloor + 1)\right] = \frac{\lfloor n/l \rfloor + 1}{n} \frac{1}{\lfloor n/l \rfloor + 1} D(k(\lfloor n/l \rfloor + 1),\, l(\lfloor n/l \rfloor + 1))$$

$$\rightarrow \frac{1}{l}\, \eta(k, l) \quad \text{as } n \rightarrow \infty \quad \text{a.s.}$$

Hence, (6.1) holds for all positive rational x. A similar argument applies to (6.2).

To treat irrational x, we apply Theorem 5.1a to deduce that $\gamma(x + y) - \gamma(x)$ is decreasing in x through rational x for each rational y. Hence, γ is nondecreasing and concave restricted to the rationals. Since γ is nondecreasing overall, γ is nondecreasing and concave, and thus continuous, overall. Hence, the limits (6.1) and (6.2) extend to irrational x. (Note that $n^{-1} D(\lfloor nx \rfloor, n)$ is sandwiched between corresponding averages for rationals that converge. This implies the existence of convergent subsequences as $n \rightarrow \infty$. The continuity of γ on the rationals then implies that all limits of convergent subsequences converge to a common limit, implying convergence for the full sequence.)

To see that $\gamma(x + y) - \gamma(x) \geq y$, so that γ is strictly increasing, use the fact that

$$D(\lfloor (x + y) n \rfloor, n) \geq D(\lfloor xn \rfloor, n) + \sum_{i = \lfloor xn \rfloor + 1}^{i = \lfloor (x + y) n \rfloor} V(i, n) .$$

By considering only paths through $(2k, 2k)$ for all k, $1 \leq k \leq \min\{xn, n\}$, we easily obtain the lower bound in (6.3). To see that $\gamma(x) \geq 1$ for all $x > 0$, note that, for all x, $D(\lfloor xn \rfloor, n) > D(1, n)$ for all n sufficiently large. Since $D(k, n) \stackrel{d}{=} D(n, k)$ for all n and k, we see that

$$\gamma(x) = \lim_{n \rightarrow \infty} n^{-1} D(\lfloor xn \rfloor, n) = \lim_{n \rightarrow \infty} \frac{\lfloor xn \rfloor}{n} \frac{1}{\lfloor xn \rfloor} D(\lfloor xn \rfloor, \lfloor xn \rfloor / x)$$

$$= \lim_{n \rightarrow \infty} \frac{\lfloor xn \rfloor}{n} \frac{1}{\lfloor xn \rfloor} D(\lfloor xn \rfloor / x, \lfloor xn \rfloor) = x\gamma(x^{-1}) . \quad \blacksquare$$

To illustrate the path-independence bound, in (6.4)–(6.6) suppose that the service times have an exponential distribution as in Theorem 6.1. Then

$$Ee^{\theta[V(1,1)-1]} = (1-\theta)^{-1}e^{-\theta} \quad \text{and} \quad h(a) = a - \log(1+a) \ . \tag{6.8}$$

From (6.5) and (6.8), we obtain a^* by solving

$$(1+x)[a - \log(1+a)] = (1+x)\log(1+x) - x\log x \ ,$$

which for the case $x = 1$ is

$$a^* - \log(1+a^*) = \log 2 \ ,$$

yielding $a^* = 1.68$ and $\overline{\lim}_{n\to\infty} n^{-1}D(n,n) \leq 5.36$. From Theorem 6.1 we see that this is indeed an upper bound, which seems to be not a terrible approximation. Evidently, there is enough dependence among the paths to reduce this estimate by a factor of 0.746.

Example 6.1. It is possible that the infimum in (6.5) is not attained as an equality. For example, suppose that $V(i,j)$ is Bernoulli, assuming the values 0 and 2 each with probability 1/2. Then $h(a) = [(1+a)\log(1+a) + (1-a)\log(1-a)]/2$, $0 \leq a < 1$, and $h(a) = \infty$ for $a \geq 1$. For $x = 1$, $\lim_{a\to 1-} h(a) = \log 2$, so that $a^*(1) = 1$, which yields $\gamma(1) \leq 4$.

7. More Properties of the Departure RBM

We established the strong approximation in Section 4 in order to deduce more about the departure times of customer k_n from queue n when $k_n \to \infty$ as $n \to \infty$. We now establish a limit for the way the departure RBM components $\hat{D}_k(1)$ grow as $k \to \infty$ that enables us to conclude that the average of the first $\lfloor xn^a \rfloor$ interdeparture times from queue n after the first departure is of order $n^{(1-a)/2}$ for any $x > 0$ and any a satisfying $0 < a < 1$. The limit is obtained by applying the subadditive ergodic theorem once more.

Theorem 7.1. Let $\hat{D} = f(\hat{B})$. Then there exists a deterministic nondecreasing concave function $\hat{\gamma}(x)$ such that

$$\lim_{n \to \infty} n^{-1} \hat{D}_{\lfloor xn \rfloor}(n) = \hat{\gamma}(x) \quad \text{a.s.} \tag{7.1}$$

so that

$$n^{-\frac{1}{2}} \hat{D}_{\lfloor xn \rfloor}(1) \Rightarrow \hat{\gamma}(x) \quad \text{as } n \to \infty \tag{7.2}$$

for each $x > 0$.

Proof. As in the proof of Theorem 6.3, we apply the subadditive ergodic theorem on p. 277 of Liggett (1985). We first establish the limit for $n^{-1}\hat{D}_{jn}(kn)$ for j and k integer. We let $-X_{0,n} = \hat{D}_{jn}(n)$ and $-X_{m,n}$ be $-X_{0,n-m}$ applied to the shifted process $B_i'(t) = B_{i+km}(t + lm) - B_{i+km}(lm)$. With this definition, X is subadditive, i.e., $X_{0n} \leq X_{0m} + X_{m,n}$ for $0 \leq m \leq n$, and X satisfies the other conditions of the subadditive ergodic theorem, except possibly for the bound. To establish the bound, we consider a related discrete problem. We consider the $kn \times ln$ integer lattice. We associate with the point (i,j) in this lattice the random variable

$$W(i,j) = \sup_{j-1 \leq t < j} |B_i(t) - B_i(j-1)| . \tag{7.3}$$

It is easy to see that

$$\hat{D}_{jn}(kn) \leq \sup_{\pi \in \Pi(jn,kn)} \{ \sum_{(i,j) \in \pi} W(i,j) \} + W(jn,kn) . \tag{7.4}$$

For each path π, the random variables $W(i,j)$ for $(i,j) \in \pi$ are i.i.d. Moreover, for different paths, the partial sums are associated. Hence, we have a path-independence bound for the right side of (7.4) paralleling Theorem 6.2. Using the known tail behavior of $W(i,j)$ in (7.3), we have the required bound for the subadditive ergodic theorem. Hence, $n^{-1} \hat{D}_{jn}(kn)$ converges a.s. to a proper limit as $n \to \infty$. As in the proof of Theorem 6.3, we use this result to deduce that (7.1) holds for each rational x. We then apply Corollary 2 to Theorem 5 to deduce that

$\hat{\gamma}(x+y) - \hat{\gamma}(x)$ is decreasing in x through rational x for each rational y. Hence $\hat{\gamma}$ is nondecreasing and concave restricted to the rationals. Since $\hat{\gamma}$ is nondecreasing overall, $\hat{\gamma}$ is nondecreasing and concave overall. Hence, (7.1) extends to irrational x. Since $n^{-1/2}\,\hat{D}\lfloor xn \rfloor(1) \stackrel{d}{=} n^{-1}\hat{D}_{\lfloor xn \rfloor}(n)$ for all n, we obtain (7.2) directly from (7.1). ■

We now apply Theorem 7.1 to obtain a limit for the average of the first $\lfloor n^a \rfloor$ departure times for $0 < a < 1$. The following shows that this average is asymptotically of order $n^{(1-a)/2}$.

Theorem 7.2. Under the assumptions of Theorem 4.1,

$$\frac{D(\lfloor xn^a \rfloor, n) - D(1,n)}{n^{(1+a)/2}} \Rightarrow \hat{\gamma}(x)$$

for $\hat{\gamma}(x)$ in Theorem 7.1 and $0 < a < 1$.

Proof. Note that

$$\frac{D(\lfloor xn^a \rfloor, n) - D(1,n)}{n^{(1+a)/2}} = \left[\frac{\sqrt{n}\hat{D}_{\lfloor xn^a \rfloor}(1)}{n^{(1+a)/2}}\right] + \left[\frac{D(\lfloor xn^a \rfloor, n) - n - \sqrt{n}\hat{D}_{\lfloor xn^a \rfloor}(1)}{n^{(1+a)/2}}\right] - \left[\frac{D(1,n) - n}{n^{(1+a)/2}}\right]. \tag{7.5}$$

By (7.2) in Theorem 7.1, the first term on the right in (7.5) converges in probability to $\hat{\gamma}(x)$. By Theorem 4.1, the second term on the right converges in probability to 0. By (1.1), the third term on the right converges to 0. ■

8. Concluding Remarks

There are several stones left unturned. First, it would be nice to identify the hydrodynamic limit $\gamma(x)$ in (6.1) for non-exponential service-time distributions. It would be nice to determine how this limit depends on the service-time distribution. We have established that there is an invariance principle for $a < 1$. We conjecture that the limit when $a = 1$ depends on the service-time distribution beyond its first two moments. It would also be nice to establish a refined distributional limit, i.e., a weak convergence limit for $n^{-\alpha}(D(\lfloor xn \rfloor, n) - \gamma(x)n)$ as $n \to \infty$.

Second, it would be nice to know more about the departure RBM $f(\hat{B})$ (Section 3). In Remark 3.3 we noted that $\hat{\Delta}_1 = \hat{D}_2 - \hat{D}_1$ is a reflecting BM, so that $E[\hat{\Delta}_1(1)] = 2\sigma/\sqrt{\pi}$. Moreover, by Theorem 5.2, $\hat{\Delta}_k(1)$ is stochastically decreasing as k increases. However, it would be nice to know the joint distribution, or at least the means, of $(\hat{\Delta}_1(1), \ldots, \hat{\Delta}_k(1))$. Moreover, it would be nice to know the hydrodynamic limit $\hat{\gamma}(x)$ for the Departure RBM in (7.1) and (7.2). By Theorem 7.2, it determines the limit in Theorem 1.1 for $0 < a < 1$.

Finally, an old open problem is the limiting behavior of the stationary departure process from n queues as $n \to \infty$. Here we assume that the service times $V(i, j)$ are i.i.d. for $i \geq 1$ and $j \geq 2$ while the service times $V(i, 1)$ are i.i.d. (or just stationary and ergodic) with $E\,V(1, 1) > E\,V(1, 2)$, so that the departure process from the first queue corresponds to an external arrival process with mean interarrival time greater than the subsequence mean service times. For the case in which $V(1, 2)$ is exponentially distributed, but $V(1, 1)$ is not, it is widely believed that the stationary departure process from queue n is asymptotically Poisson as $n \to \infty$. A corresponding result for infinite-server queues was established by Vere-Jones (1968).

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